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# HOMOLOGICAL STABILITY FOR $O_{n,n}$ OVER A LOCAL RING

STANISŁAW BETLEY

ABSTRACT. Let $R$ be a local ring, $V^{2n}$ a free module over $R$ of rank $2n$ and $q$ a bilinear form on $V^{2n}$ which has in some basis the matrix $\left|\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right|$. Let $O_{n,n}$ be the group of automorphisms of $V^{2n}$ which preserve $q$. We prove the following theorem: if $n$ is big enough with respect to $k$ then the inclusion homomorphism $i\colon O_{n,n} \to O_{n+1,n+1}$ induces an isomorphism $i_*\colon H_k(O_{n,n}; Z) \to H_k(O_{n+1,n+1}; Z)$.

**Introduction.** Let $R$ be a local ring, $V^{2n}$ a free module over $R$ of rank $2n$ and $q$ a bilinear form on $V^{2n}$ which has in some basis the matrix $\left|\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right|$ (we will call such a form hyperbolic). Let $O_{n,n}$ be the group of automorphisms of $V^{2n}$ which preserve $q$. The main goal of this paper is to prove the following theorem (Theorem 3.1):

THEOREM. *If $n$ is big enough with respect to $k$ then the inclusion homomorphism $i\colon O_{n,n} \to O_{n+1,n+1}$ induces an isomorphism $i_*$ from $H_k(O_{n,n}; Z)$ to $H_k(O_{n+1,n+1}; Z)$.*

Questions of this type became interesting in the early seventies; at that time people were mostly interested in the study of the stability of the homology groups of $GL_n(R)$, $R$ was a field or a local ring (see [**Bo**, **Q**, **W**]). In [**Ch**] Charney proved a stability theorem for $\mathrm{Aut}(W)$ where $W$ was a projective module over a Dedekind ring. Dwyer in [**D**] extended Charney's result to twisted systems of coefficients. The most general results in this direction were obtained by Van der Kallen in [**K**] (see also [**M**]). He proved the twisted homological stability for $GL_n(R)$ over rings which satisfy the Bass stable range condition $SR_{s\dim+2}(R)$ (see [**Ba**]). Later on in the seventies there appeared the notion of hermitian $K$-theory and questions about the homological stabilization of various orthogonal groups began to be interesting. In [**V1**] Vogtmann proved stability for $O_{n,n}$ over infinite fields and in [**V2**] she generalized her result to all fields but $Z_2$. Her [**V2**] was the fundamental motivation for this paper.

Vogtmann's [**V3**] is the other result in this direction. She proved there a homological stability theorem for $O_n(F)$, where $F$ is an arbitrary field of finite "Pythagoras number" and $O_n(F)$ means the ordinary orthogonal group (the group of the identity form).

Homological stability problems are closely related to algebraic $K$-theory. If one defines nonstable $K$-groups as $\pi_i(BGL_n^+(R))$ (see [**S**]), then the stability for the homology groups of $GL_n(R)$ often implies the stability for $K$-groups. Moreover,

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using homological stability theorems and relations between homotopy and homology groups one can better understand the $K$-groups (see for example [**Ch**, Corollary 3.7, 3.8]). So the following paper is the one of many possible approaches to better understanding the orthogonal $K$-groups.

The methods used in this paper are generally the same as ones used in the above-mentioned papers on the homological stability for various types of groups. In the first two chapters we will find a highly connected simplicial complex on which our group acts with well-known isotropy subgroups. Then in §III we will use spectral sequences associated with this action to obtain the desired homological result.

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**1.**

1.1. DEFINITION. Let $X$ be a simplicial complex spanned by the set of vertices $\{v_i\}_{i\in I}$. We say $X$ satisfies the condition $E_k$ if for every subset $\{v_j\}_{j=1}^{n}$, $n \le k$, there exists a vertex $v \in \{v_i\}_{i\in I}$ such that $X$ contains the join of $v$ and the subcomplex spanned by the set $\{v_j\}_{j=1}^{n}$.

Let now $n_1(s)$ be the number of vertices in the 2nd barycentric subdivision of $\Delta^s$ and $n_2(s)$ be the number of vertices in the first barycentric subdivision of $\partial\Delta^s$. Then we have

1.2. PROPOSITION. *If $X$ is a simplicial complex which satisfies $E_k$ for $k \ge n_1(s) + n_2(s)$ then $X$ is $s$-connected.*

PROOF. Let $f\colon S^s \to |X|$ ($|X|$ is a geometric realization of $X$). We can assume that $f$ is a simplicial map from some barycentric subdivision of $\partial\Delta^{s+1}$ to $X$. If $f$ is a map from the $m$th barycentric subdivision, $m > 1$, then we will construct a homotopy of this map to some map from the $(m-1)$st barycentric subdivision of $\partial\Delta^{s+1}$ to $X$. This means that up to homotopy we can assume that $f$ is a simplicial map from the first barycentric subdivision of $\partial\Delta^{s+1}$ to $X$. But it is easy to see that $n_2(s+1) < n_1(s) + n_2(s) \le k$ so $f$ is homotopic to 0—we can form a cone in $X$ over the image of $f$.

*Construction of the homotopy.*

*Step* I. *Triangulation of $S^s \times I$.* Let $Y$ be any simplicial complex, $Y^1$ and $Y^2$ be its first and second barycentric subdivisions. We will describe a new triangulation of $Y \times I$ such that $Y \times \{0\}$ is $Y^2$ and $Y \times \{1\}$ is $Y^1$. If $\{\delta\}$ is the set of simplices in $Y$ then we can consider $Y^1$ as the simplicial complex spanned by the set $\{\delta\}$ as the set of vertices. The collection $\delta_1, \delta_2, \dots, \delta_k$ span a $(k-1)$-simplex iff after some reordering $\delta_1 \subsetneq \delta_2 \subsetneq \cdots \subsetneq \delta_k$. Similarly we can consider $Y^2$ as the simplicial complex spanned by the set of ordered tuples $(\delta_1 \subsetneq \delta_2 \subsetneq \cdots \subsetneq \delta_p)$ where the collection of tuples $(\delta_{1,1} \subsetneq \cdots \subsetneq \delta_{1,p_1}), \dots, (\delta_{r,1} \subsetneq \cdots \subsetneq \delta_{r,p_r})$ span an $(r-1)$-simplex iff after some reordering we have $(\delta_{1,1} \subsetneq \cdots \subsetneq \delta_{1,p_1}) \subsetneq \cdots \subsetneq (\delta_{r,1} \subsetneq \cdots \subsetneq \delta_{r,p_r})$ and the inclusion of tuples means that the smaller tuple is a subcollection of the larger one. Now the triangulation of $Y \times I$ is given by the following rules:

(a) the set of vertices is given by $\{\text{vertices of } Y^1\} \cup \{\text{vertices of } Y^2\}$;

(b) the collection $\delta_1, \dots, \delta_k, (\delta_{1,1} \subsetneq \cdots \subsetneq \delta_{1,p_1}), \dots, (\delta_{r,1} \subsetneq \cdots \subsetneq \delta_{r,p_r})$ span a simplex iff after some reordering $\delta_1 \subsetneq \cdots \subsetneq \delta_k$ and $(\delta_{1,1} \subsetneq \cdots \subsetneq \delta_{1,p_1}) \subsetneq \cdots \subsetneq (\delta_{r,1} \subsetneq \cdots \subsetneq \delta_{r,p_r})$ and $\delta_{r,p_r} \subset \delta_1$.

This amounts to saying that every vertex $\delta$ from $Y^1$ is joined by an edge only with those vertices from $Y^2$ which belong to its own barycentric subdivision. Higher dimensional simplices are given by the rule that if a boundary of some simplex belongs to the triangulation of $Y \times I$ so does this simplex.

*Step* II. *Homotopy.* Let us call now the $(m-2)$nd barycentric subdivision of $\partial\Delta^{s+1}$ as $Y$, $(m-1)$st as $Y^1$ and $m$th as $Y^2$. Using the notation from Step I we have only to extend given $f\colon Y^2 \to X$ to $F\colon Y \times I \to X$ where $Y \times I$ is triangulated accordingly to Step I. So we have only to define $F$ on the set $\{\delta\}$ of simplices from $Y$ considered now as vertices from $Y^1$.

We will do that by induction with respect to the dimension of $\delta$:

(a) if $\dim(\delta) = 0$ put $F(\delta) = f(\delta)$;

(b) if $\dim(\delta) > 0$, say $\dim(\delta) = p$, then we have already defined $F$ on the set $A$ consisting of vertices from the second barycentric subdivision of $\delta$ considered as vertices in $Y^2 = Y \times \{0\}$ and on the set $B$ of vertices from the first subdivision of $\partial\delta$ considered as vertices from $Y^1 = Y \times \{1\}$. We have $\mathrm{card}(A \cup B) = n_1(p) + n_2(p)$ so by the hypothesis there is a vertex $v_\delta$ in $X$ which forms a cone in $X$ over the subcomplex spanned by $F(A \cup B)$. Put $F(\delta) = v_\delta$. That way we get the desired extension.

So we have finished the proof that $\pi_s(X) = 0$. But if $t < s$ then $n_1(t) < n_1(s)$ and $n_2(t) < n_2(s)$ so $k > n_1(t) + n_2(t)$ and by the same construction we can get $\pi_t(X) = 0$. This completes the proof of 1.2.

1.3. PROPOSITION. *Assume that $Y, Y', K$ are simplicial complexes, $Y'$ is a subcomplex of $Y$ and $K$ is finite. Assume that for every finite subcomplex $X$ of $Y$ the inclusion $X \hookrightarrow Y$ is homotopic to some simplicial map $h\colon X^{\mathrm{tr}} \to Y'$. This means that there is a simplicial map $H\colon X \times I \to Y$ together with a triangulation of $X \times I$ such that $X \times \{0\} = X$, $X \times \{1\} = X^{\mathrm{tr}}$, $H|_{X\times\{1\}} = h$, $H|_{X\times\{0\}} = \mathrm{id}_X$. The complex $X \times I$ with this triangulation will be called $\overline{X \times I}$. Let $f\colon K \to Y$ be a simplicial map. Then $f$ is homotopic to $f'\colon \mathrm{sd}(K) \to Y'$ where $\mathrm{sd}(K)$ is some barycentric subdivision of $K$.*

PROOF. Let $X = f(K)$. Then $X$ is a finite subcomplex of $Y$. We have maps

$$|K| \times I \xrightarrow[|f|\times\mathrm{id}]{} |X| \times I \xrightarrow[|H|]{} |Y|.$$

Then there exists a simplicial map $G\colon \mathrm{sd}(K \times I) \to \overline{X \times I}$ such that

(a) $G|_{K\times\{0\}}$ is homotopic to $f$;

(b) $G(K \times \{1\}) \subset X^{\mathrm{tr}}$.

(The map $G$ is any simplicial approximation to $|f| \times \mathrm{id}$; for details see [**Sp**, Chapter 3].) Then we can take as $f'$ the map $H$ composed with $G|_{K\times\{1\}}$. Obviously $f'$ is homotopic to $f$ and its image is in $Y'$.

**2.** Let $R$ be a local ring and $\mathfrak{m}$ be a maximal ideal in $R$. Let $H^{2n}$ be the standard $2n$-dimensional hyperbolic module over $R$ with the standard basis $e_1, \dots, e_n$, $f_1, \dots, f_n$. Let $q$ be the standard form on $H^{2n}$ given by $q(e_i, e_j) = q(f_i, f_j) = 0$, $q(e_i, f_j) = \delta_{i,j}$.

2.1. DEFINITION. (a) $Y(H^{2n})$ is a simplicial complex obtained by the following rules:

(1) the set of vertices consists of all unimodular isotropic vectors in $H^{2n}$;

(2) the set $(v_1, \dots, v_k)$ forms a $(k-1)$-simplex in $Y(H^{2n})$ iff span $(v_1, \dots, v_k)$ is a $k$-dimensional, isotropic direct summand of $H^{2n}$.

(b) $Y_k(H^{2n})$ is the $k$-skeleton of $Y(H^{2n})$.

2.2. REMARK. Let $v \in H^{2n}$, $v = (a_1, \dots, a_n; b_1, \dots, b_n)$ in the basis $e_1, \dots, e_n$, $f_1, \dots, f_n$. Then $v$ is unimodular isotropic iff $2 \cdot \sum_{i=1}^{n} a_i b_i = 0$ and at least one of $a_i$'s or $b_i$'s is invertible.

Now we have to introduce some notation. Assume that $1 \leq i \leq 2n$. Then if $i \leq n$, the $(n+1)$th coordinate in $H^{2n}$ will be called the dual coordinate to the $i$th one and we will denote it by $i^*$. If $n < i \leq 2n$ then the $(i-n)$th coordinate will be called dual and will be denoted $i^*$.

2.3. REMARK. If $q(v_1, v_2) = 0$, the $i$th coordinate of $v_1$ is invertible and the $i^*$th coordinate of $v_2$ is invertible then there is $j \neq i$, $j \neq i^*$ such that the $j$th coordinate of $v_1$ is invertible and the $j^*$th coordinate of $v_2$ is invertible.

PROOF. This is obvious by the formula for $q$ and by the following property of $R$: if $a \in R \backslash \mathfrak{m}$ and $b \in \mathfrak{m}$ then $a + b \in R \setminus \mathfrak{m}$.

2.4. REMARK. Let $n \geq s+1$ and let $(v_1, \dots, v_s)$ form an $(s-1)$-simplex in $Y(H^{2n})$. Then there is a natural way of finding a vector $v$ such that $(v, v_1, \dots, v_s)$ form an $s$-simplex. This vector will be called the special vector for $(v_1, \dots, v_s)$.

The idea of finding a special vector is very simple: first we have to find a so-called upper-triangular basis for span $(v_1, \dots, v_s)$ and then we have to take the most simple unimodular vector which is isotropic and perpendicular to all vectors of the new basis.

*The description of the special vector for* $(v_1, \dots, v_s)$. We can assume that the following conditions are satisfied modulo the action of the permutation group $\Sigma_{2n}$ which permutes the coordinates of $H^{2n}$ (we also have to use 2.3):

$a/v_1$ has an invertible element on the first coordinate;

$b/v_i^{i-1}$ has an invertible element on the $i$th coordinate where $1 \leq i \leq s$, $0 \leq j < i$,

$$v_i^0 = v_i, \quad v_i^j = v_i^{j-1} - a_i^{j-1} v_j^{j-1}$$

and $a_i^{j-1}$'s are elements of $R$ such that $v_i^k$ has 0 on the coordinates $1, \dots, k$. Then span$(v_1, \dots, v_s)$ =span$(v_1, v_2^1, \dots, v_s^{s-1})$ and in the coordinate system for $H^{2n}$ we have

$$\begin{aligned} v_1 &= (a_1, \dots) \\ v_2^1 &= (0, a_2, \dots) \\ &\vdots \\ v_s^{s-1} &= (0, \dots, 0, a_s, \dots) \end{aligned}$$

where $a_1, \dots, a_s$ are invertible ($\{v_1, v_2^1, \dots, v_s^{s-1}\}$ form the upper-triangular basis for span $(v_1, \dots, v_s)$).

Without losing generality we can assume that $a_1 = a_2 = \cdots = a_s = 1$. Let now $j \neq 1$, $j \neq 2, \dots, j \neq s$, $j \neq 1^*, \dots, j \neq s^*$. Then we can easily find

$$(*) \qquad v = \Big(0, \dots, 0, \underset{j}{1}, 0, \dots, 0; \underset{1^*}{b_1}, \dots, \underset{s^*}{b_s}, 0, \dots, 0\Big)$$

such that span $(v, v_1, v_2^1, \ldots, v_s^{s-1})$ is isotropic and has rank $s+1$ (so $(v, v_1, \ldots, v_s)$ is an $s$-simplex in $Y(H^{2n})$). If

$$\begin{aligned} v_1 &= (1, x_2^0, \ldots, x_n^0; y_1^0, \ldots, y_n^0) \\ v_2^1 &= (0, 1, x_3^1, \ldots, x_n^1; y_1^1, \ldots, y_n^1) \\ &\vdots \\ v_s^{s-1} &= (0, \ldots, 0, 1, x_{s+1}^{s-1}, \ldots, x_n^{s-1}; y_1^{s-1}, \ldots, y_n^{s-1}) \end{aligned}$$

then

$$\begin{aligned} b_s &= -y_j^{s-1} \\ b_{s-1} &= -(x_s^{s-2} b_s + y_j^{s-2}) \quad \text{and so on.} \end{aligned}$$

The vector $v$ from $(*)$ is the desired special vector for $(v_1, \ldots, v_s)$. If we consider the general case without assuming (a) and (b), then we obtain the vectors $v_1, v_2^1, \ldots, v_s^{s-1}$ which have invertible elements on some coordinates $i_1, \ldots, i_s$ respectively. Let $j$ be such a number that $j \neq i_1, \ldots, j \neq i_s, j \neq i_1^*, \ldots, j \neq i_s^*$. Then the special vector $v$ for $(v_1, \ldots, v_s)$ will be of the type $(*)$, but it will have $b_1$ on the coordinate $i_1^*, b_2$ on the coordinate $i_2^*$ and so on. The way of finding a special vector for $(v_1, \ldots, v_s)$ we will call the special construction for $(v_1, \ldots, v_s)$.

2.5. REMARK. (a) The special vector for $(v_1, \ldots, v_s)$ has exactly $s+1$ coordinates not equal to 0.

(b) In the construction described in 2.4, $v_i^{i-1} = v_i - B$ where $B$ is some linear combination of $v_1, \ldots, v_{i-1}$, $i = 2, \ldots, s$.

(c) The special vector for $(v_1, \ldots, v_s)$ is fully described by

(i) ordering of the set $\{v_1, \ldots, v_s\}$;

(ii) a choice of $i_1, \ldots, i_s, j$.

2.6. DEFINITION. (a) Let $Z_m(H^{2n})$ be a subcomplex of $Y(H^{2n})$ which is spanned by vertices having no more than $m+1$ coordinates not equal to 0.

(b) Let $Z_{m,k}(H^{2n}) = Z_m(H^{2n}) \cap Y_k(H^{2n})$.

(c) Let $Z_m(H^{2n})_{i_1, \ldots, i_k}$ be the subcomplex of $Z_m(H^{2n})$ spanned by vertices which have 0 on the coordinates $i_1, \ldots, i_k$.

2.7. LEMMA. *Let $(v_1, \ldots, v_s)$ form an $(s-1)$-simplex in $Y(H^{2n})$, $n \geq s+1$ and let in some special construction for $(v_1, \ldots, v_k)$, $k < s$, $v_j^{j-1}$ has chosen invertible element on the coordinate $i_j$, $1 \leq j \leq k$. Then we can find $v \in Z_s(H^{2n})_{i_1, \ldots, i_k}$ such that $(v, v_1, \ldots, v_s)$ form an $s$-simplex in $Y(H^{2n})$.*

PROOF. We have only to perform the special construction for $(v_1, \ldots, v_s)$, which extends the given special construction for $(v_1, \ldots, v_k)$. It means that $v_j^{j-1}$'s and $i_j$'s in the construction for $(v_1, \ldots, v_s)$, $j \leq k$, should be the same as in the given construction for $(v_1, \ldots, v_k)$.

2.8. LEMMA. *Let $K$ be a finite subcomplex of $Y(H^{2n})$ of dimension $m$ having fewer than $t$ vertices. Let $A = (v_1, \ldots, v_k)$ be a $(k-1)$-simplex in $K$, $k \leq m+1$ and let $i_1, \ldots, i_k$ be as in the previous lemma. Then there exist functions $\alpha(t, m)$, $\beta(t)$ such that, if $n > \alpha(t, m)$ then the inclusion $K \hookrightarrow Y(H^{2n})$ is homotopic by an inclusion $M \hookrightarrow Y(H^{2n})$ (this means that $M$ is homotopy equivalent to $K \times I$) to an inclusion $K' \hookrightarrow Y(H^{2n})$ where $M, K'$ have the following properties:*

(a) vertices$(M) \setminus$ (vertices$(M) \cap$ vertices$(K)) \subset Z_{m+1}(H^{2n})_{i_1,\dots,i_k}$;
(b) $A \notin K'$;
(c) *$M$ has fewer than $\beta(t)$ vertices.*

PROOF. The general idea of the proof is very simple. In order to find $M$ it would be enough to build a cone in $Y(H^{2n})$ over $\mathrm{star}_K A$, which has its new vertex in $Z_{m+1}(H^{2n})_{i_1,\dots,i_k}$ (we consider here $\mathrm{star}_K A$ as the set of all simplices of $K$ which contain $A$). Unfortunately it is not always possible to find such a vertex. So the construction will go this way: at the beginning we will homotope $K$ to a finite complex $L$ such that $\mathrm{star}_L A$ is contained in a subcomplex generated by $A$ and $Z_{m+1}(H^{2n})_{i_1,\dots,i_k}$ and has a small number of vertices with respect to $n$. Then almost every vertex given by the chosen special construction for $A$ will form a cone over $\mathrm{star}_L A$—the only requirement is that it will have 1 on some coordinate $s$, such that the vertices from $\mathrm{star}_L A$ which are not in $A$ have 0 on the coordinates $s$ and $s^*$. This is easy to obtain if only $n$ is big enough.

Consider now $K$ as an abstract simplicial complex. In Step I of the proof we will build complexes $M \supset K$, $K' \subset M$ having the desired properties (b) and (c) and a small number of vertices. In Step II we will construct an extension of the inclusion $K \hookrightarrow Y(H^{2n})$ to $M$ in such a way that we obtain property (a).

*Step* I. A construction will go by some decreasing induction with respect to the dimension of $A$.

(i) If $\dim(A) = m$ then we have to add one new vertex $v_A$ to $K$ and form $M$ by the following rule: if $\delta \in K$ then $\delta \cup \{v_A\}$ forms a simplex in $M$ iff $\delta \subset A$; $K'$ is a subcomplex of $M$ obtained by the following rule: if $\delta \in K$ then $\delta \cup \{v_A\}$ forms a simplex in $K'$ iff $\delta \subset A$ but $\delta \neq A$.

(ii) If $\dim(A) = k-1 < m$ then we will construct a sequence of complexes $K = M_m = K'_m, M_{m-1}, K'_{m-1}, \dots, M_{k-1}, K'_{k-1}, M, K'$ such that $M_i$ is contained in $M_{i-1}$ and the inclusion is a homotopy equivalence, $K'_i$ is contained in $M_i$ and is homotopy equivalent to it, $k-1 \le i \le m$, $M_{k-1}$ is contained in $M$ and is homotopy equivalent to it, $K' \subset M$ and the inclusion is a homotopy equivalence. The inductive step of getting $M_{i-1}$ from $M_i$ is given by the following construction:

Let $B_1, \dots, B_p$ be the set of the $i$-dimensional simplices in $\mathrm{star}_K A$ considered now as the simplices in $K'_i$. We have $i \ge k$ so we can find $\overline{M}_{B_1}, K'_{B_1}$ which are the solution of the same problem for $B_1 \subset K'_i$. Let $M_{B_1} = M_i \cup_{K'_i} \overline{M}_{B_1}$. Generally for every $1 \le j \le p$ we can find $\overline{M}_{B_j}, K'_{B_j}$ which are the solution for $B_j$ contained in $K'_{B_{j-1}}$. Let now $M_{B_j} = M_{B_{j-1}} \cup_{K'_{B_{j-1}}} \overline{M}_{B_j}$. Then we define $M_{i-1} = M_{B_p}$, $K'_{i-1} = K'_{B_p}$. Repeating the described above construction we finally get $M_{k-1}$ and $K'_{k-1}$. The complex $M$ is obtained from $M_{k-1}$ by adding one new vertex $v_A$ and the simplices in $M$ are given by the rule: if $\delta \in M_{k-1}$ then $\delta \cup \{v_A\}$ forms a simplex in $M$ iff $\delta \subset \delta'$ and $A \subset \delta'$, $\delta' \in K'_{k-1}$. The complex $K'$ is a subcomplex of $M$ obtained from $K'_{k-1}$ by the rule: if $\delta \in K'_{k-1}$ then $\delta \cup \{v_A\}$ forms a simplex in $K'$ iff $\delta \subset \delta'$, $A \subset \delta'$, $A \not\subset \delta$.

REMARK. (a) It is obvious that we have got $M$ from $K$ by adding only a finite number of vertices (bounded by some function $\beta(t)$).

(b) All inclusions $M_i \hookrightarrow M_{i-1}$, $M_{k-1} \hookrightarrow M$, $K'_i \hookrightarrow M_i$, $K' \hookrightarrow M$ are homotopy equivalences by the construction—on every step we have built only cones over stars of some simplices, which are contractible.

*Step* II. We have only to extend the inclusion $K \hookrightarrow Y(H^{2n})$ to $M$ in such a way, that all new vertices will belong to $Z_{m+1}(H^{2n})_{i_1,\dots,i_k}$. We will construct this extension by induction as in Step I. We will call the desired inclusion as $i\colon M \hookrightarrow Y(H^{2n})$.

(i) If $k-1=m$ let $i(v_A)=$ the special vector for $(v_1,\dots,v_k)$ which is obtained by the chosen special construction and which does not belong to $K$.

(ii) If $k-1<m$ then by induction we can get an extension of $i$ to $M_{k-1}$. But $M_{k-1}$ is obtained from $K$ by adding less than $\beta(t)$ vertices. Let $w_A$ be the special vector for $A$ obtained by the given special construction with 1 on some coordinate $p$, such that every vertex $w$ in $\mathrm{star}_{i(M_{k-1})}A \setminus A$ has 0 on the coordinates $p$ and $p^*$ and $w_A$ does not belong to $K$. It is possible to find such a number $p$ if we define $\alpha(t,m)=(m+2)\cdot(\beta(t)-1)+k$ because we know that all vertices $w$ in $\mathrm{star}_{i(M_{k-1})}A \setminus A$ have only $m+1$ coordinates not equal to 0.

Now we can put $i(v_A)=w_A$. It is clear that properties (a), (b) and (c) are satisfied.

2.9. PROPOSITION. *Let $i\colon L \hookrightarrow Y(H^{2n})$ be an embedding of some simplicial complex of dimension $m$ into $Y(H^{2n})$. If $n$ is big enough then there is a triangulation of $L\times I$ and a simplicial map $I\colon L\times I \to Y(H^{2n})$ such that*

(a) $L\times\{0\}=L$;

(b) $I(L\times\{1\})\subset Z_{m+1}(H^{2n})$;

(c) $I|_{L\times\{0\}}=i$.

PROOF. We have to construct a homotopy of $i$ to some simplicial map $i^1: L^{\mathrm{tr}} \to Z_{m+1}(H^{2n})$, where $L^{\mathrm{tr}}$ denotes some subdivision of $L$. We will construct this homotopy step by step over skeletons using Lemma 2.8. We consider $L$ as a subcomplex of $Y(H^{2n})$.

*Step* I. For any vertex $w$ in $L$ choose a vertex $v$ by some special construction for $w$ ($v$ has only two coordinates not equal to 0). This gives us the homotopy over the 0-skeleton of $L$.

*Step* II. Assume that we have already constructed our homotopy over the $k$-skeleton of $L$. Let $A$ be a $(k+1)$-simplex of $L$. Then we have already built our homotopy over $\partial A$. Let $K$ be a subcomplex of $Y(H^{2n})$ consisting of $A$ and the given homotopy over $\partial A$. Then $K$ satisfies the hypothesis of Lemma 2.8 and all vertices of $K$ are in $Z_{m+1}(H^{2n})$ excepting those which belong to $A$. Thus we can apply 2.8 to $A\in K$. That way we get a homotopy $M$ of $K$ to $K'$ such that $A\notin K'$ and all vertices in $M\setminus A$ are in $Z_{m+1}(H^{2n})$. Let $B_1^k,\dots,B_{k+2}^k,B_1^{k-1},\dots,B_{k+2}^0$ be the set of all faces of $A$, where the upper index denotes the dimension of the simplex and the lower one means some ordering of the set of equal-dimensional simplices. Then there exist $K'_{B_1^k}$ and $M_{B_1^k}$ given by 2.8 as the solution for $B_1^k\in K'$. Generally there is $K'_{B_r^p}$ and $M_{B_r^p}$ given by 2.8 as the solution for $B_r^p\in K'_{B_{r-1}^p}$ if $r\neq 1$ or $B_r^p\in K_{B_s^{p+1}}$, where $s=\binom{k+2}{p+2}$. Let us call now $M_{B_{k+2}^0}$ as $M''$ and $K'_{B_{k+2}^0}$ as $K''$. Then we have an embedding of $M''$ into $Y(H^{2n})$ which gives us a homotopy between $K$ and $K''$ and $K''\subset Z_{m+1}(H^{2n})$. Now we can use $M''$ as the desired triangulation of $A\times I$ and the inclusion $M''\hookrightarrow Y(H^{2n})$ as the extension of $i$ over $A$. Obviously both of them are well defined and we can perform this construction separately over each $(k+1)$-simplex of $L$. Thus finally we get an extension of our homotopy on the $(k+1)$-skeleton of $L$.

2.10. REMARK. If $k(m+1) < n$ then $Z_m(H^{2n})$ satisfies the condition $E_k$.

PROOF. Every vertex of $Z_m(H^{2n})$ has at most $m+1$ coordinates not equal to 0 so if $k(m+1) < n$ then for every $v_1, \dots, v_k \in Z_m(H^{2n})$ there exists a vector $v$ from the canonical basis for $H^{2n}$ such that the join of $v$ and the subcomplex spanned by $v_1, \dots, v_k$ is contained in $Z_m(H^{2n})$.

2.11. THEOREM. *If $n$ is big enough with respect to $m$ then $Y(H^{2n})$ is $m$-connected.*

PROOF. If $k \le m$ and $n$ is sufficiently big with respect to $m$ then by 1.3 and 2.9 every $f\colon S^k \to Y(H^{2n})$ is homotopic to $f'\colon S^k \to Z_{m+1}(H^{2n}) \subset Y(H^{2n})$. But then by 2.10 and 1.2 $f'$ is homotopic to 0.

2.12. REMARK. In fact we have proved that if $m \ll n$ and $k \ll n$ then $Z_{m,k}(H^{2n})$ and $Y_k(H^{2n})$ have the homotopy type of a wedge of spheres.

We will need some more spherical simplicial complexes. Let us consider the following type of simplicial complexes.

2.13. DEFINITION. (i) $Y^i(H^{2n})$ is a simplicial complex obtained by the following rules:

(a) vertices are given by ordered $i$-tuples $(v_1, \dots, v_i)$ such that in the canonical basis for $H^{2n}$ we have

$$\begin{aligned} v_1 &= (1, 0, \dots, 0, x^1_{i+1}, \dots, x^1_n; y^1_1, \dots, y^1_n) \\ v_2 &= (0, 1, 0, \dots, 0, x^2_{i+1}, \dots, x^2_n; y^2_1, \dots, y^2_n) \\ &\vdots \\ v_i &= (0, \dots, 0, 1, x^i_{i+1}, \dots, x^i_n; y^i_1, \dots, y^i_n) \end{aligned}$$

and span $(v_1, \dots, v_i)$ is an $i$-dimensional, isotropic direct summand of $H^{2n}$;

(b) $k$ vertices form a $(k-1)$-simplex in $Y^i(H^{2n})$ iff all vectors involved in them span a $ki$-dimensional, isotropic direct summand of $H^{2n}$.

(ii) $Y^i_k(H^{2n})$ denotes the $k$-skeleton of $Y^i(H^{2n})$.

We are going to prove that if $n$ is big enough with respect to $k$ and $i$, then $Y^i_k(H^{2n})$ has a homotopy type of a wedge of $k$-spheres. The proof will be similar to the one for $Y_k(H^{2n})$. We will give all the details of the proof in the case $i = 1$. The case $i > 1$ will follow from the previous one.

First of all we have to justify that the special construction still exists in the case of $Y^1(H^{2n})$.

2.14. LEMMA. *Let $(v_1, \dots, v_s) \in Y(H^{2n})$ , $v_1 \in Y^1(H^{2n})$ and $n > s+3$. Then there exists a vector $v \in Y^1(H^{2n})$ such that $(v, v_1, \dots, v_s)$ span a rank $(s+1)$ direct summand of $H^{2n}$ and $v$ is of the type*

(a) $(1, 0, \dots, 0, 1, 0, \dots; 0, \dots, c_1, \dots, c_s, 0, \dots, 0)$, *$c_1$ is on the coordinate $j_1, \dots,$ $c_s$ is on the coordinate $j_s$, 1 is on the $k$th and first coordinates—there are only $s+2$ coordinates not equal to 0 and $1 \ne k$, $1 \ne k^*$, for every $1 \le t \le s$, $1 \ne j_t \ne 1^*$, $k \ne j_t \ne k^*$ and for every $t_1$ and $t_2$, $j_{t_1} \ne j_{t_2}$, $j_{t_1} \ne j^*_{t_2}$; or*

(b) $(1, 0, \dots, 0, 1, 0, \dots; a, 0, \dots, 0, -a, c_1, \dots, c_{s-1}, 0, \dots, 0)$, $a \ne 0$*—there are only $s+3$ coordinates not equal to 0, the assumptions on the coordinates $k$, $j_1, \dots, j_{s-1}$ hold as in* (a) *and $-a$ is on the coordinate $k^*$.*

PROOF. The proof of this lemma is similar to the one of the existence of the special construction for $Y(H^{2n})$ so we will use the notation from 2.4.

*Note.* We can assume that $1^* \notin \{i_1, \dots, i_s\}$ because all vectors of the type $v_i^j$ are perpendicular to $v_1$ (use 2.3). The difference between 2.4 and this lemma is that now there are two possibilities:

I. It is possible to find suitable indices $i_1, \dots, i_s$ such that $1 \in \{i_1, \dots, i_s\}$. Then we can easily find the desired vector of the type (a), precisely by the same way as in 2.4.

II. There is $1 \le j \le s$ such that $i_j$ must be equal to 1. We will find a vector $v'$ which is perpendicular to the vectors $v_s^{s-1}, \dots, v_j^{j-1}$ and

$$v' = (1 \dots 1 \dots ; a, \dots, -a, d_1, \dots, d_{s-j}, \dots)$$

with the other coordinates equal to 0, 1 is on the first and $m$th coordinates, $-a$ is on the coordinate $m^*$, $d_t$ is on the coordinate $j_t^*$ for $1 \le t \le s-j$, $m \ne i_1, \dots, m \ne i_s$, $m \ne i_1^*, \dots, m \ne i_s^*$ and $\{j_1, \dots, j_{s-j}\} = \{i_{j+1}, \dots, i_s\}$.

Let now

$$\begin{aligned} v_s^{s-1} &= (x_1^{s-1}, \dots, x_{2n}^{s-1}) \\ v_{s-1}^{s-2} &= (x_1^{s-2}, \dots, x_{2n}^{s-2}) \\ &\vdots \\ v_1^0 &= (x_1^0, \dots, x_{2n}^0). \end{aligned}$$

By the construction we can assume that

(i) $x_1^{j-1} = 1$, $x_i^{j-1}$ is not invertible for $i > 1$;

(ii) $x_q^p = 0$ if $q \in \{i_1, \dots, i_p\}$.

So we have only to solve the system of equations (for simplicity we assume that, as in 2.4, the coordinate $i_k$ of $v_k^{k-1}$ is equal to 1):

$$\begin{aligned}
&x_{n+1}^{j-1} + x_{m^*}^{j-1} + a - a x_m^{j-1} + d_1 x_{j_1^*}^{j-1} + \dots + d_{s-j} x_{j_{s-j}^*}^{j-1} = 0 && (q(v', v_j^{j-1}) = 0) \\
&x_{n+1}^{s-1} + x_{m^*}^{s-1} - a x_m^{s-1} + d_{s-j} = 0 && (q(v', v_s^{s-1}) = 0) \\
&x_{n+1}^{s-2} + x_{m^*}^{s-2} - a x_m^{s-2} + d_{s-j} x_{j_{s-j}^*}^{s-2} + d_{s-j-1} = 0 && (q(v', v_{s-1}^{s-2}) = 0) \\
&\vdots \\
&x_{n+1}^{j} + x_{m^*}^{j} - a x_m^{j} + d_1 + \sum_{p>1} d_p x_{j_p^*}^{j} = 0 && (q(v', v_{j+1}^{j}) = 0).
\end{aligned}$$

We can solve this system of equations step by step:

(i) from the first equation we can find $a$ because $x_m^{j-1}$ is not invertible so $1 - x_m^{j-1}$ is invertible;

(ii) from the second equation we can find $d_{s-j}$ because the coefficient at $d_{s-j}$ in the expression for $a$ is not invertible;

(iii) if we have already had $a, d_{s-j}, \dots, d_{s-k}$, we can find $d_{s-k-1}$ from the next equation because the coefficients at $d_{s-k-1}$ in the expression for $a, d_{s-j}, \dots, d_{s-k}$ are not invertible.

That way we will get $v'$. Now it is easy to obtain $v$ from $v'$. We have only to add suitable elements of $R$ on the coordinates $i_1^*, \dots, i_{j-1}^*$ in the description for $v'$ (compare 2.4).

2.15. REMARK. Let ${}^jY^1(H^{2n})$ be a subcomplex of $Y(H^{2n})$ spanned by vertices which have 1 on the coordinate $j$ ($Y^1(H^{2n}) = {}^1Y^1(H^{2n})$). Then 2.14 holds if we replace $Y^1(H^{2n})$ by ${}^jY^1(H^{2n})$ and $v \in {}^jY^1(H^{2n})$.

2.16. REMARK. The special construction is still fully described by

(i) a choice of coordinates $i_1, \dots, i_s$;

(ii) a choice of a one additional coordinate $k$ such that for every $1 \le j \le s$, $k \ne i_j$ and $k \ne i_j^*$.

2.17. DEFINITION. (i) $Y^1_{a,b}(H^{2n})$ is the subcomplex of $Y^1(H^{2n})$ spanned by vertices of the types (a) and (b) from 2.14;

(ii) $Y^1_{a,b,m}(H^{2n})$ is the subcomplex of $Y^1_{a,b}(H^{2n})$ spanned by vertices which have at most $m+3$ coordinates not equal to 0;

(iii) $Y^1_{a,m}(H^{2n})$ is the subcomplex of $Y^1(H^{2n})$ spanned by vertices of the type (a) which have at most $m+2$ coordinates not equal to 0;

(iv) $\overline{Y^1_{a,b,m}(H^{2n})}$ is the subcomplex of $Y^1_{a,b,m}(H^{2n})$ consisting of simplices $(v_1, \dots, v_s)$ such that for every $1 \le j \le s$, $v_j$ has some coordinate $i_j, i_j \ne 1$, $i_j \ne 1^*$ which has the following property: $v_j$ has an invertible element on $i_j$th coordinate and if $k \ne j$ then $v_k$ has 0 on the coordinates $i_j$ and $i_j^*$ (if $(v_1, \dots, v_s)$ is a simplex in $\overline{Y^1_{a,b}(H^{2n})}$ then there is a special vector $v$ for $(v_1, \dots, v_s)$ which belongs to $Y^1_{a,s}(H^{2n})$);

$v/Z_{i_1,\dots,i_k}$ denotes the subcomplex of $Z$ spanned by vertices which have 0 on the coordinates $i_1, \dots, i_k$ if $1 \notin \{i_1, \dots, i_k\}$ or on the coordinates $i_1, \dots, i_{j-1}, i_{j+1}, \dots, i_k$ if $i_j = 1$, where $Z$ is any of the described above complexes or $Y^1(H^{2n})$.

2.18. LEMMA. *Let $(v_1, \dots, v_s)$ form an $(s-1)$-simplex in $Y^1(H^{2n})$, $n > s+2$ and suppose that in some special construction for $(v_1, \dots, v_k)$, $v_j^{j-1}$ has a chosen invertible element on $i_j$th coordinate, $1 \le j \le k$ (notation from 2.4). Then we can find $v \in Y^1_{a,b,s}(H^{2n})_{i_1,\dots,i_k}$ such that $(v, v_1, \dots, v_s)$ form an $s$-simplex in $Y(H^{2n})$.*

PROOF. It is obvious by the properties of the special construction.

2.19. LEMMA. *Let $K \subset Y^1(H^{2n})$ be a finite subcomplex of dimension $m$ having less than $t$ vertices and $L \subset K$ be a subcomplex contained in $\overline{Y^1_{a,b,p}(H^{2n})}$. Let $A = (v_1, \dots, v_k)$ be a simplex in $K$ and let $i_1, \dots, i_k$ be as in the previous lemma. Then there is a function $\alpha'(t,m,p)$ such that if $n > \alpha'(t,m,p)$, the inclusion $K \hookrightarrow Y^1(H^{2n})$ is homotopic via an inclusion $M \hookrightarrow Y^1(H^{2n})$ to an inclusion $K' \hookrightarrow \overline{Y^1_{a,b,s}H^{2n}}$ for some $s < n$ and*

(a) *$M \setminus K \cup L \subset \overline{Y^1_{a,b,s}(H^{2n})}$ where $M \setminus K \cup L$ denotes the subcomplex generated by $L$ and vertices from $M \setminus K$;*

(b) vertices$(M) \setminus$ (vertices$(M)$ $\cap$ vertices$(K)) \subset Y^1_{a,b,s}(H^{2n})_{i_1,\dots,i_k}$;

(c) $A \notin K'$;

(d) *$M$ has fewer than $\beta(t)$ vertices.*

PROOF. The idea of the proof is the same as in 2.8. We would like to build a cone over $\mathrm{star}_K A$ in such a way, that the new vertex is in $Y^1_{a,b,s}(H^{2n})$ and the conditions (a) and (b) are satisfied. Obviously it is not always possible to find such a good vertex, so we will first rebuild $\mathrm{star}_K A$ and at the very end we will form a cone over the new star. We will use the notation from the proof of 2.8, especially from Step I.

Again it is enough to extend the inclusion $K \hookrightarrow Y^1(H^{2n})$ to an inclusion $i\colon M \hookrightarrow Y^1(H^{2n})$ such that (a) and (b) are satisfied. Let

$$\alpha'(t,m,p) = t(p+3) + (\beta(t)+m+3)\cdot(\beta(t)-1).$$

We will construct our homotopy by induction as in 2.8.

(1) If $k-1=m$ then let $i(v_A)$ be equal to the special vector for $(v_1,\dots,v_k)$ obtained by the given special construction with 1 on some coordinate such that (a) is satisfied and $i(v_A)$ is not in $K$.

(2) If $k-1<m$ then by induction we have an extension of our inclusion to $M_{k-1}$. But now we have to do that extension by a special way: all new vertices $w_1,\dots,w_q$ which we have added to $K$ during the construction should satisfy the following condition:

(*) for every $1\le j\le q$, $w_j$ has 1 on some coordinate $k_j$, such that all $w_i$'s for $i\ne j$ have 0 on the coordinates $k_j$ and $k_j^*$ ($k_j$'s are not equal to 1 or $1^*$).

Then by (b) and (*) $\operatorname{span}(v_1,\dots,v_k,w_1,\dots,w_q)$ is a free direct summand of $H^{2n}$ of the rank $q+k$. So now we have to find $w_A$ such that $q(w_A,v_i)=0$, $q(w_A,w_j)=0$ for $1\le i\le k$, $1\le j\le q$ and $\operatorname{span}(w_A,v_1,\dots,v_k,w_1,\dots,w_q)$ is a direct summand of $H^{2n}$ of the rank $q+k+1$, where $w_A\in Y^1_{a,b,s}(H^{2n})$.

We will obtain $w_A$ by the same way as we used in the proof of 2.14. We again have two possibilities:

I. $1\notin\{i_1,\dots,i_k\}$. In this case we can easily obtain $w_A$ which belongs to $Y^1_{a,q-k}(H^{2n})$ by the same technique as $v$ in 2.4.

II. $1\in\{i_1,\dots,i_k\}$. In this case we will get

$$w_A = (1,\dots,1,\dots;a,\dots,-a,c_1,\dots,c_{k-1},d_1,\dots,d_q,\dots)$$

where $c_j$ is on the coordinate $i_j$, $d_i$ is on the coordinate $k_i$, 1 is on the first and some other coordinate $r$ such that $r\ne i_j$, $r\ne i_j^*$, $r\ne k_i$, $r\ne k_i^*$ for $1\le j\le k$ and $1\le i\le q$, $a$ is on the coordinate $1^*$, $-a$ is on the coordinate $r^*$ and the other coordinates in $w_A$ are 0. We can get such $w_A$ by solving the following system of equations:

$$\begin{array}{ll} q(v_1,w_A)=0 & q(w_1,w_A)=0\\ \vdots & \vdots\\ q(v_k,w_A)=0 & q(w_q,w_A)=0. \end{array}$$

We are able to solve it precisely for the same reason as the solution existed in the proof of 2.14. Obviously we can find $w_A$ this way that $w_A\notin i(M_{k-1})$.

So now we have only to define $i(v_A)=w_A$. That way we get an extension of $i$.

REMARK. (1) The condition (*) can be easily obtained because we have to add to $K$ less than $\beta(t)$ vertices during the construction of $M_{k-1}$ and the image of every new vertex under $i$ has at most $\beta(t)+m+2$ coordinates not equal to 0.

(2) The conditions (b), (c) and (d) are obviously satisfied. In order to obtain (a) we have to assume that, if $v_1,\dots,v_j$ are vertices of $A$ and the simplex $(v_1,\dots,v_j)\in Y^1_{a,b,p}(H^{2n})$, then we can find such $r$ that all $v_i$'s, $1\le i\le j$ have 0 on the coordinates $r$ and $r^*$. But this is still possible because $n>\alpha'(t,m,p)$ and $j<t$ and every $v_i$ has at most $p+3$ coordinates not equal to 0, $i=1,\dots,j$.

2.20. PROPOSITION. *Let $i\colon N \hookrightarrow Y^1(H^{2n})$ be an inclusion of some simplicial complex into $Y^1(H^{2n})$, $\dim(N) = m$. If $n$ is big enough then there exist a triangulation of $N \times I$ and a simplicial map $I\colon N \times I \to Y^1(H^{2n})$ which have the following properties:*

(a) $N \times \{0\} = N$;

(b) $I(N \times \{1\}) \subset \overline{Y^1_{a,b,p}(H^{2n})}$ *for some* $p < n$;

(c) $I|_{N\times\{0\}} = i$.

PROOF. We will construct a triangulation and $I$ inductively over skeletons using 2.19. This proof is the same as the proof of 2.9 so we will skip some details.

(1) The situation is very easy over the 0-skeleton of $N$—for every vertex $v \in N$ we have to find a vector $w$, which is a special vector for $v$ given by any special construction.

(2) Assume that we have already constructed the triangulation and the map $I$ over the $k$-skeleton of $N$. Let $A$ be a $(k+1)$-simplex of $N$ and $K$ be a subcomplex of $Y^1(H^{2n})$ consisting of $A$ and the constructed homotopy over $\partial A$. Let $K \setminus A \subset \overline{Y^1_{a,b,r}(H^{2n})}$. We can apply to these $K, A$ and $r$ Lemma 2.19 using as $L$ the subcomplex of $K$ spanned by vertices($K$)\vertices($A$)—that way we get rid of $A$. Then we should apply 2.19 again and again for all faces of $A$ (for details see the proof of 2.9). By (b) of 2.19 we know that this way we obtain the desired triangulation of $A \times I$ and the extension of $I$ over $A$; in particular all new vertices are in $Y^1_{a,b,p}(H^{2n})$ and all new simplices are in $\overline{Y^1_{a,b,p}(H^{2n})}$ for some $p < n$.

2.21. PROPOSITION. *Let $i\colon N \hookrightarrow \overline{Y^1_{a,b}(H^{2n})}$ be an inclusion of some simplicial complex of dimension $m$ into $\overline{Y^1_{a,b}(H^{2n})}$. Then, if $n$ is big enough, there is a triangulation of $N \times I$ and a simplicial map $I\colon N \times I \to Y^1(H^{2n})$ such that*

(a) $N \times \{0\} = N$;

(b) $I(N \times \{1\}) \subset Y^1_{a,m+1}(H^{2n})$;

(c) $I|_{N\times\{0\}} = i$.

PROOF. The proof of this proposition is exactly the same as the proof of the previous one if we use one obvious observation: if the complex $K$ from 2.19 is in $\overline{Y^1_{a,b}(H^{2n})}$ then $M$ can be chosen that way, that $M \setminus K \subset Y^1_{a,m+1}(H^{2n}) \cap \overline{Y^1_{a,b}(H^{2n})}$. This observation follows directly from the definition of $\overline{Y^1_{a,b}(H^{2n})}$.

2.22. REMARK. If $k(m+2) < n$ then $Y^1_{a,m}(H^{2n})$ satisfies the condition $E_k$.

PROOF. The same as the proof of 2.10.

Now we are ready to prove

2.23. THEOREM. *If $n$ is large enough with respect to $k$ then $Y^1_k(H^{2n})$ is $(k-1)$-connected.*

PROOF. Let $f\colon S^m \to Y^1(H^{2n})$ be any map, $m < k$. We can assume that $f$ is a simplicial map from some barycentric subdivision of $\partial\Delta^{m+1}$ to $Y^1_k(H^{2n})$. By Propositions 2.20, 2.21 and 1.3 we obtain that the obvious map

$$\pi_m(Y^1_{a,m+1,k}(H^{2n})) \to \pi_m(Y^1_k(H^{2n}))$$

is an epimorphism, where $Y^1_{a,m+1,k}(H^{2n})$ is the $k$-skeleton of $Y^1_{a,m+1}(H^{2n})$. But by 1.2 and 2.22 we know that $\pi_m(Y^1_{a,m+1,k}(H^{2n})) = 0$ if only $m < k$ and $n$ is big enough. This finishes the proof.

2.24. THEOREM. *If $n$ is large enough with respect to $k$ and $i$ then $Y^i_k(H^{2n})$ is $(k-1)$-connected.*

PROOF. This theorem follows precisely the same way as in the case $i = 1$ which has been already proved. Let us point out the four important steps in the proof:

(1) If $A = (w_1, \ldots, w_k)$ is a $(k-1)$-simplex in $Y^i(H^{2n})$ then for every $1 \leq s \leq k$, $w_s = (v^s_1, \ldots, v^s_i)$ where $v^s_j \in {}^jY^1(H^{2n})$ (see 2.15) and $v^s_j$ has 0 on the coordinates $1, \ldots, j-1,\ j+1, \ldots, i$. In order to build a cone over $A$ it is enough to find $w = (v_1, \ldots, v_i)$ such that $v_j \in {}^jY^1(H^{2n})$, $j = 1, \ldots, i$ and $v_1$ forms a cone over $(v^1_1, \ldots, v^1_i, v^2_1, \ldots, v^k_1, \ldots, v^k_i)$, $v_2$ forms a cone over $(v^1_1, \ldots, v^k_i, v_1)$ and so on. But we can certainly do that by 2.14—the fact that $v_j$ should have 0 on the coordinates $1, \ldots, j-1, j+1, \ldots, i$ is easy to obtain using 2.3: every vertex of the type $v^q_p$ used in the proof of the existence of the special construction must have an invertible element not only on the coordinates $1^*, \ldots, i^*$.

So we have the special construction for $Y^i(H^{2n})$ and two types of $i$-tuples:

(a) vectors in a tuple belong to ${}^jY^1_{a,b}(H^{2n})$;

(b) vectors in a tuple belong to ${}^jY^1_a(H^{2n})$.

This gives us two subcomplexes of $Y^i(H^{2n})$, namely $Y^i_{a,b}(H^{2n})$ and $Y^i_a(H^{2n})$ (and a lot of their variations as previously).

(2) Lemma 2.19 is valid for $Y^i(H^{2n})$, where $\overline{Y^i_{a,b}(H^{2n})}$ denotes the subcomplex of $Y^i_{a,b}(H^{2n})$ defined as follows: $(w_1, \ldots, w_k)$ is a $(k-1)$-simplex in $\overline{Y^i_{a,b}(H^{2n})}$ iff $(v^1_1, \ldots, v^1_i, v^2_1, \ldots, v^k_i)$ has the same property as $(v_1, \ldots, v_s)$ in the definition of $\overline{Y^1_{a,b}(H^{2n})}$ with the additional condition $i_j \neq 1, \ldots, i_j \neq i,\ i_j \neq 1^*, \ldots, i_j \neq i^*$ (for the notation see 2.17). In the proof of 2.19 we built only cones over stars of some simplices. But now we can do the same—in order to do that we have only to have the special construction and the finiteness of the complex (now we should assume $n > i((\beta(t) + m + 3 + i) \cdot (\beta(t) - 1) + t(p + 3)))$.

(3) Propositions 2.20 and 2.21 hold for $Y^i(H^{2n})$—it is obvious because in order to prove them we needed only 2.19.

(4) If $kmi < n$ then $Y^i_{a,m}(H^{2n})$ satisfies the condition $E_k$, where $Y^i_{a,m}(H^{2n})$ is a subcomplex of $Y^i_a(H^{2n})$ such that every vector in every tuple in $Y^i_{a,m}(H^{2n})$ has at most $m + 2$ coordinates not equal to 0.

**3.** Now we are able to prove the main theorem.

3.1. THEOREM. *Let $R$ be a commutative local ring, $q_n$ be the hyperbolic form on the free $R$-module $H^{2n}$ of the rank $2n$. If $n$ is large enough with respect to $k$ then the inclusion homomorphism $O(H^{2n}, q_n) \hookrightarrow O(H^{2n+2}, q_{n+1})$ induces an isomorphism on homology $i_*\colon H_k(O(H^{2n}, q_n); Z) \to H_k(O(H^{2n+2}, q_{n+1}); Z)$.*

The proof will be similar to the Vogtmann's proof for fields (see [**V2**]). We will denote $O(H^{2n}, q_n)$ by $O_n$, the complex $Y(H^{2n})$ by $Y$, the $k$-skeleton of $Y$ by $Y_k$, the complex $Y^i(H^{2n})$ by $Y^i$ and the $k$-skeleton of $Y^i$ by $Y^i_k$.

For the fixed $k$ we have a filtration of $Y_k$ by skeletons $\varnothing = Y_{-1} \subset Y_0 \subset \cdots \subset Y_{k-1} \subset Y_k$. If $k \ll n$ then $Y_i$ is $(i-1)$-spherical for all $0 \leq i \leq k$ and we have an exact sequence

$$(*)\qquad \begin{aligned} 0 &\to H_k(Y_k) \to H_k(Y_k, Y_{k-1}) \to H_{k-1}(Y_{k-1}, Y_{k-2}) \to \cdots \\ &\to H_1(Y_1, Y_0) \to H_0(Y_0) \to Z \to 0. \end{aligned}$$

The group $O_n$ acts on $Y_k$ for every $k \leq n$. The filtration, which we have described above, is invariant under the action of $O_n$.

It is easy to see that for $k \geq i$ we have

$$H_i(Y_i, Y_{i-1}) = \bigoplus_{\text{all } i\text{-simplices } A} Z_A$$

where $Z_A$ is isomorphic to $Z$ for every $A$. The group $O_n$ acts on $H_i(Y_i, Y_{i-1})$ by permuting the copies of $Z_A$ with a possibility of changing an orientation, given by the action of $O_n$ on $Y_{i-1}$.

Let $0 \to C_{k+2} \to C_{k+1} \to \cdots \to C_0 \to 0$ denote the exact sequence $(*)$. Let $E_*$ be a free $Z[O_n]$-resolution of $Z$. Then we can form the double complex $E_* \otimes C_*$ and we get two spectral sequences associated with it. Because $E_*$ is free and $C_*$ is exact, both of them will converge to 0. One of them has

$$\begin{aligned} E^1_{p,q} &= H_q(E_* \otimes C_p) = H_q(O_n; C_p) = H_q\left(O_n; \bigoplus_{(p-1)\text{-simplices}} Z_A\right) \\ &= H_q(O_n; Z[O_n]) \bigotimes_{Z[S_{p,n}]} Z_P = H_q(S_{p,n}; Z_P), \end{aligned}$$

where $S_{p,n}$ is the stabilizer in $O_n$ of the frame $(e_1, \ldots, e_p)$ (we are using the fact that $O_n$ acts transitively on $p$-frames, see [**R**] for details), $Z_P = Z_{(e_1,\ldots,e_p)}$.

Thus we have the following lemma.

3.2. LEMMA. *There is a spectral sequence converging to* 0 *with*

$$E^1_{p,q} = H_q(S_{p,n}; Z_P) \quad \textit{for } 0 \leq p \leq k.$$

If we consider $O_n$ acting from the left on $H^{2n}$ then $S_{p,n}$ is the subgroup of $O_n$ consisting of matrices of the form:

$$\begin{vmatrix} S & * & * & * \\ 0 & A & * & B \\ 0 & 0 & S^{-t} & 0 \\ 0 & C & * & D \end{vmatrix}$$

where $S \in \Sigma_p$—the group of the permutations of $p$ elements and $\left|\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right| \in O_{n-p}$.

Consider now the action of $F_{p,n}$ on $Y^p$ from the right, where $F_{p,n}$ is a subgroup of $S_{p,n}$ consisting of matrices which have $S = I$. This action is simplicial, transitive and it preserves the filtration of $Y^p$ by skeletons. By the same consideration as before Lemma 3.2 we will get (always for very large $n$):

3.3. LEMMA. *There is a spectral sequence converging to* 0 *with*

$$E^1_{0,t} = H_t(F_{p,n}; Z) \quad \textit{and} \quad E^1_{s,t} = H_t(R_{s-1,p,n}; Z_B),$$

*where $R_{i,p,n}$ is the stabilizer in $F_{p,n}$ of the $i$-simplex $B = (w_1, \ldots, w_{i+1}) \in Y^p$, $w_j = (e_1 + e_{(j-1)\cdot p+1}, \ldots, e_p + e_{jp})$, $1 \leq j \leq i+1$, $Z_B = Z$ as a group and the action of $R_{i,p,n}$ on $Z_B$ is given by the action of $R_{i,p,n}$ on $\partial B$.*

We will need relative versions of Lemmas 3.2 and 3.3 (for details see [**V2**]).

3.4. LEMMA. *There is a spectral sequence converging to* 0 *with*

$$E^1_{p,q} = H_q(S_{p,n+1}, S_{p,n}; Z_P).$$

3.5. LEMMA. *There is a spectral sequence converging to* 0 *with*

$$E^1_{0,t} = H_t(F_{p,n+1}, F_{p,n}; Z)$$

*and* $E^1_{s,t} = H_t(R_{s-1,p,n+1}, R_{s-1,p,n}; Z_B)$ *for* $0 < s < k$.

Now we can prove the main theorem. We will proceed by induction on $k$. Let us consider the following two conditions:

$(\mathrm{a})_k$ $H_i(O_{n+1}, O_n) = 0$ for $i < k$, $n$ large enough;

$(\mathrm{b})_k$ $H_i(F_{p,n+1}, F_{p,n}) = 0$ for $i < k$, $n$ large enough with respect to $k$ and $p$ (when we do not write any system of coefficients it means that we consider homology with a trivial one).

We know $S_{1,n} = F_{1,n}$ by the definition. Let

$$j_*\colon H_k(O_{n-1}, O_{n-2}) \to H_k(O_n, O_{n-1})$$

be the map given by the composition of the maps

$$H_k(O_{n-1}, O_{n-2}) \xrightarrow{d} H_k(F_{1,n}, F_{1,n-1})$$

and $H_k(S_{1,n}, S_{1,n-1}) \xrightarrow{d'} H_k(O_n, O_{n-1})$, where $d$ is the $d_1$-map of the spectral sequence of 3.5 and $d'$ is the $d_1$-map of the spectral sequence of 3.4. Then in order to prove that $H_k(O_{n+1}, O_n) = 0$ it is enough to show that $j_*\colon H_k(O_{n-1}, O_{n-2}) \to H_k(O_n, O_{n-1})$ is onto and then to use the diagram chase of the following diagram:

$$\begin{array}{ccccccc}
 & & H_k(O_{n-1}, O_{n-2}) & \longrightarrow & H_{k-1}(O_{n-2}) & & \\
 & & \downarrow{\scriptstyle j_*} & & \downarrow & & \\
H_k(O_n) & \longrightarrow & H_k(O_n, O_{n-1}) & \longrightarrow & H_{k-1}(O_{n-1}) & \longrightarrow & H_{k-1}(O_n) \\
\downarrow & & \downarrow{\scriptstyle j_*} & & & & \\
H_k(O_{n+1}) & \longrightarrow & H_k(O_{n+1}, O_n) & & & &
\end{array}$$

So we need only to show that $j_*$ is onto.

3.6. LEMMA. *Let* $d'\colon H_k(S_{1,n}, S_{1,n-1}) \to H_k(O_n, O_{n-1})$ *be the* $d_1$*-map of the spectral sequence of* 3.4. *If* $n$ *is large enough then* $d'$ *is onto.*

PROOF. It is enough to show that $d'$ is the only possible nonzero derivation which goes to $H_k(O_n, O_{n-1})$ so it is enough to show that

$$(**) \qquad E^1_{p,q} = 0 \quad \text{for } p+q \le k+1,\ p \ge 2 \quad (\text{so } q \le k-1).$$

Consider the extensions

$$\begin{array}{ccccccccc}
1 & \longrightarrow & F_{p,n-1} & \longrightarrow & S_{p,n-1} & \longrightarrow & \Sigma_p & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & F_{p,n} & \longrightarrow & S_{p,n} & \longrightarrow & \Sigma_p & \longrightarrow & 1
\end{array}$$

The mapping cone spectral sequence (with coefficients in $Z_P$) of that diagram has

$$E^2_{u,v} = H_u(\Sigma_p; H_v(F_{p,n}, F_{p,n-1}; Z_P)) \Rightarrow H_{u+v}(S_{p,n}, S_{p,n-1}; Z_P).$$

But $F_{p,n}$ acts trivially on $Z_P$ so by our induction hypothesis $(\mathrm{b})_{k-1}$ we obtain that for $n$ large enough, the condition $(**)$ is satisfied.

3.7. LEMMA. *Let $d\colon H_k(O_{n-1},O_{n-2}) \to H_k(F_{1,n},F_{1,n-1})$ be the $d_1$-map of the spectral sequence from* 3.5. *Then for $n$ large enough $d$ is an epimorphism.*

PROOF. We need only to show that $E^1_{s,t} = 0$ for $s+t \le k+1$, $t \ge 2$, in the spectral sequence from 3.5. We know that $E^1_{s,t} = H_t(R_{s-1,p,n}, R_{s-1,p,n-1}; Z_B)$. The group $R_{s,p,n}$ consists of the matrices of $F_{p,n}$ having the form

$$n\left\{\begin{array}{r} p\{ \\ sp\{ \\ \\ \\ \\ \\ \end{array}\right.\left|\begin{array}{cccccc} I & * & 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 & 0 & 0 \\ 0 & * & A & 0 & 0 & B \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & * & * & * & X^{-t} & * \\ 0 & * & C & 0 & 0 & D \end{array}\right| \quad \text{where } \left|\begin{array}{cc} A & B \\ C & D \end{array}\right| \in O_{n-sp-p}.$$

The projection map

$$R_{s,p,n} \to G_{s,p} = \left|\begin{array}{cccccc} I & * & 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & * & X^{-t} & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{array}\right|$$

has kernel equal to $F_{sp,n-p}$. Then the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & F_{sp,n-p-1} & \longrightarrow & R_{s,p,n-1} & \longrightarrow & G_{s,p} & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \| & & \\ 1 & \longrightarrow & F_{sp,n-p} & \longrightarrow & R_{s,p,n} & \longrightarrow & G_{s,p} & \longrightarrow & 1 \end{array}$$

gives us a spectral sequence with

$$\begin{aligned} E^2_{u,v} &= H_u(G_{s,p}; H_v(F_{sp,n-p}, F_{sp,n-p-1}; Z_B)) \\ &\Rightarrow H_{u+v}(R_{s,p,n}, R_{s,p,n-1}; Z_B). \end{aligned}$$

As previously $F_{sp,n-p}$ acts trivially on $Z_B$ and we can use the induction hypothesis $(\mathrm{b})_{k-1}$ if only $n$ is large enough with respect to $k, p, s$.

Thus the proof of the main theorem is almost done. We have proved that $(\mathrm{b})_{k-1}$ and $(\mathrm{a})_{k-1}$ imply $(\mathrm{a})_k$. But using the spectral sequence form 3.5 it is easy to see that $(\mathrm{b})_{k-1}$, $(\mathrm{a})_{k-1}$ and $(\mathrm{a})_k$ imply $(\mathrm{b})_k$ and that observation finishes the proof.

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WYDZIAŁ MATEMATYKI UW, INSTYTUT MATEMATYKI, PKIN IXP., 00-901WARSZAWA, POLAND